

ACCESSIBLE POINTS ROTATE AS PRIME ENDS IN BACKWARD OR FORWARD TIME

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ABSTRACT. Let X be a closed invariant subset of the half-open annulus \mathbb{A} such that $\mathbb{A} \setminus X$ is homeomorphic to \mathbb{A} . We prove that either the rotation number of all forward semi-orbits of accessible points of X are well-defined and equal to the prime end rotation number or the same is true for all backward semi-orbits of accessible points of X .

1. INTRODUCTION

Every open connected and simply connected subset U of the Riemann sphere which misses at least two points is conformally equivalent to $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by the Riemann mapping theorem. In general, the conformal map $\phi: \mathbb{D} \rightarrow U$ cannot be extended even continuously because the topology of the boundary of U can be very wild. In fact, ϕ can be extended to $\partial\mathbb{D}$ if and only if ∂U is locally connected and the extension is injective iff ∂U is a Jordan curve. Whilst proving this result, Caratheodory [C13] developed the theory of prime ends. This theory associates an ideal boundary $\mathbb{P}(U)$, the circle of prime ends of U , to every simply connected planar proper domain U such that $U \sqcup \mathbb{P}(U)$ equipped with a suitable topology is homeomorphic to $\overline{\mathbb{D}}$. More details are given in Section 3. Points in $\partial\mathbb{D}$ are identified to prime ends of U but prime ends do not correspond in general to points in ∂U .

Any homeomorphism $\varphi: \mathbb{D} \rightarrow U$ can be extended to a homeomorphism between $\overline{\mathbb{D}}$ and the prime end compactification of U , $U \cup \mathbb{P}(U)$. Evidently, if φ can be extended to a point $z \in \partial\mathbb{D}$ then we can find a curve $\gamma_z: [0, 1] \rightarrow U \cup \{\varphi(z)\}$ which lands at $\gamma_z(1) = \varphi(z)$. The converse result is true provided φ is conformal. The set of points of ∂U satisfying this property are called *accessible*. Accessible points and prime ends are very closely related.

This article focuses on the relationship between prime ends and rotation in annular dynamics. The notion of rotation number of circle homeomorphisms introduced by Poincaré was generalized first to circle degree-1 endomorphisms and then to annular and toral dynamics (see [MZ89]). A great amount of research has been conducted on retrieving information of the dynamics from the rotation number/interval/set in those settings. Given a homeomorphism of the annulus $f: \mathbb{A} \rightarrow \mathbb{A}$ isotopic to the identity and F a lift of f to the universal cover of \mathbb{A} , the *rotation interval* of an f -invariant continuum K is the set of limit points of sequences of the form $(F^{n_i}(x_i))_1 - (x_i)_1/n_i$, where n_i tends to $+\infty$, x_i belongs to the lift of K and $(\cdot)_1$ is the value of the lift of the angular coordinate in the cover. The rotation interval is, indeed, an interval which, as opposed to the circle homeomorphisms case, may be non-degenerate. Alternatively, the rotation interval can be defined as the set of values $\int_{\mathbb{A}} u d\mu$, where u is the displacement function ($u(z) = (F(\tilde{z}))_1 - (\tilde{z})_1$ for any lift \tilde{z} of z) and μ ranges over all f -invariant Borel probability measures supported on K .

Assume K is an essential annular continuum, that is, $\mathbb{A} \setminus K$ has exactly two connected components U_+ and U_- which are homotopically non-trivial. If K does not properly contain any other essential annular continuum it is called a *circloid*. Barge and Gillette [BG91] proved the following realization result for invariant circloids with empty interior (*cofrontiers*): any rational number in the rotation interval of K is realized as the rotation number of a periodic orbit in K . Koropecski [K16] has recently generalized this theorem to any circloid. Furthermore, the periodic orbit can be chosen from ∂K provided the rotation interval is non-degenerate [KP].

Using prime end theory we can compactify each annular domain U_{\pm} with a circle $\mathbb{P}(U_{\pm})$ and the result is a closed annulus. A homeomorphism which leaves U_{\pm} invariant induces homeomorphisms in $\mathbb{P}(U_{\pm})$, which are partially classified by their rotation numbers ρ_{\pm} . Matsumoto [M12] (see an alternative proof in [H16]) proved that these numbers belong to the rotation interval of X . However, even cofrontiers may

2010 *Mathematics Subject Classification.* Primary 37E30. Secondary: 37E45.

Key words and phrases. Rotation number, prime ends, accessible point.

have non-degenerate rotation intervals so one cannot expect to learn much of the dynamics within X just from the prime end rotation number.

Another illustrative example of this phenomenon are Birkhoff attractors (see [L88]). The endpoints of the rotation interval are the prime end rotation numbers ρ_+, ρ_- , and every rational number $\rho_- < a/b < \rho_+$ is realized by a periodic orbit $p_{a/b}$ in the attractor Λ . Take one of these periodic points $p_{a/b}$ and assume without loss of generality that it is hyperbolic. Its stable manifold $W^s(p_{a/b})$ meets Λ in infinitely many accessible points because $p_{a/b}$ is not accessible. Every $x \in W^s(p_{a/b}) \cap \tilde{\Lambda}$ satisfies $\lim_{n \rightarrow +\infty} (F^n(x))_1 - an/b \rightarrow 0$ so we find plenty of accessible points in $\tilde{\Lambda}$ whose forward semi-orbit has a rotational behavior different from the translation by the prime end rotation number.

Alligood and Yorke [AY92] investigated the link between accessible periodic points and the prime end rotation number in the boundary of a basin of attraction W in \mathbb{R}^2 . Under some conditions related to the hyperbolicity of the dynamics near ∂W , they proved that there is an accessible periodic point whose rotation number is equal to the prime end rotation number. They also showed that every accessible periodic point of a planar invariant domain has the same period (and the same rotation number as well). Cartwright and Littlewood [CL51] had previously proved the same result in S^2 except for the possible existence of a fixed point result of the coalescence of periodic points. Recently, Passeggi, Potrie and Sambarino [PPS] have proven the uniqueness of the rotation number of periodic points under a weaker form of accessibility in their way to showing that if the rotation interval associated to an invariant attracting cofrontier is non-degenerate the dynamics has positive topological entropy.

Assume now that X is closed and $U = \mathbb{A} \setminus X$ is connected and can be compactified with a circle of prime ends $\mathbb{P}(U)$. A homeomorphism $f: \mathbb{A} \rightarrow \mathbb{A}$ which leaves X invariant induces a circle homeomorphism \hat{f} in $\mathbb{P}(U)$. Given a fixed F lift of f , the prime end rotation number $\hat{\rho}(F, X) \in \mathbb{R}$ of F in X measures the rotation on $\mathbb{P}(U)$ and satisfies $\hat{\rho}(F, X) \bmod \mathbb{Z} = \rho(\hat{f})$. See Section 3 for full details. Suppose $\mathfrak{p} \in \mathbb{P}(U)$ has an associated accessible point $p \in X$, i.e., the principal set of \mathfrak{p} is $\{p\}$. One may expect the rotation of p under f and the rotation of \mathfrak{p} under \hat{f} to be very similar, at least, say, asymptotically equal. However, we already saw with Birkhoff attractors that $\lim_{n \rightarrow +\infty} (F^n(p))_1/n$ can be different to $\hat{\rho}(F, X)$.

The main result of this paper shows that rotation numbers of accessible points are always well-defined and equal to the prime end rotation number either for their forward semi-orbit or for their backward semi-orbit. The theorem requires relative compactness on the orbits under consideration without which rotation may not even be well-defined. A brief discussion on the hypothesis is included in Section 2.

In the case of the closed annulus $S^1 \times [0, 1]$ the compactness assumption on orbits is automatically satisfied and the main theorem of the article reads as follows:

Theorem 1. *Let $f: S^1 \times [0, 1] \rightarrow S^1 \times [0, 1]$ be a homeomorphism isotopic to the identity leaving invariant a continuum X which contains $S^1 \times \{0\}$ and does not intersect $S^1 \times \{1\}$. Suppose the complement of X is connected. Fix F a lift of f to $\mathbb{R} \times [0, 1]$, denote \tilde{X} the lift of X and $\rho = \hat{\rho}(F, X)$ the prime end rotation number of F in X . Then, one of the following statements holds:*

- *For every accessible point $p \in \tilde{X}$, $\lim_{n \rightarrow -\infty} (F^n(p))_1/n = \rho$.*
- *For every accessible point $q \in \tilde{X}$, $\lim_{n \rightarrow +\infty} (F^n(q))_1/n = \rho$.*

Linear drifts from the expected rotational behavior are only allowed either in backward time or in forward time, not simultaneously. Our discussion on the asymptotic behavior of accessible points goes a bit further. We summarize the results obtained in this work in the next theorem.

Theorem 2. *Under the hypothesis of the previous theorem:*

- (1) *There are no accessible points $p, q \in \tilde{X}$ such that*

$$\lim_{n \rightarrow -\infty} (F^n(p))_1 - n\rho = -\infty, \quad \lim_{n \rightarrow +\infty} (F^n(q))_1 - n\rho = +\infty.$$

- (2) *Suppose now that the sequences $\{(F^n(p))_1 - n\rho\}_{n \leq 0}$ and $\{(F^n(q))_1 - n\rho\}_{n \geq 0}$ are unbounded for some accessible points $p, q \in \tilde{X}$. Then,*

- (a) *both sequences $\{(F^n(q))_1\}_{n \geq 0}$ and $\{(F^n(p))_1\}_{n \leq 0}$ have the form $n\rho + o(|n|)$ and*
- (b) *if $\rho \in \mathbb{Q}$ then any limit point of the projection onto \mathbb{A} of the forward orbit of q or the backward orbit of p is periodic and its rotation number is ρ .*

Item (1) in Theorem 2 deals with the case we named “transverse” drift: the forward and backward semi-orbit drift from the expected one in different directions. This would be precisely the case of a point $x \in \tilde{X}$ with well-defined rotation number $\rho_x = \lim_{n \rightarrow \infty} (F^n(x))_1/n \in \mathbb{R}$ different from $\hat{\rho}(F, X)$.

Theorem 2.(1) implies that x is not accessible. This result is the content of Section 5. The notion of relative winding number is essential to the proof and is introduced in Section 4.

Theorem 2.(2) deals with the general case. It is proved Section 6. There, extra hypothesis (H1) and (H2) are introduced to rule out the coexistence of forward and backward semi-orbits drifting from the expected one. These hypothesis are automatically satisfied when one of the drifts is, at least, linear or the semi-orbits do not accumulate entirely in the set of periodic points with rotation ρ provided $\rho \in \mathbb{Q}$. Several examples are introduced throughout the article to illustrate the sharpness of the results, see Theorems 18 and 20.

2. SETTING

The half-open annulus $S^1 \times (-\infty, 0]$ is denoted \mathbb{A} . The one point compactification of \mathbb{A} , obtained adding the point e^- associated to the lower end of the annulus, is homeomorphic to the closed unit disk $\overline{\mathbb{D}}$. The map $\pi: \tilde{\mathbb{A}} = \mathbb{R} \times (-\infty, 0] \rightarrow \mathbb{A}$ which sends (x, y) to $(e^{2\pi i x}, y)$ is a universal covering of \mathbb{A} . We work with an orientation-preserving homeomorphism $f: \mathbb{A} \rightarrow \mathbb{A}$. Its lift to the universal cover $F: \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ is also an orientation-preserving homeomorphism. The article focus on the dynamics of an f -invariant closed set $X \subset \mathbb{A}$ that is adherent to e^- and does not meet $\partial\mathbb{A} = S^1 \times \{0\}$. In other words, $\mathbb{A} \setminus X$ is homeomorphic to $S^1 \times (-\infty, 0] = \mathbb{A}$.

The choice of the half-open annulus as workplace is not very relevant. Any dynamics in the closed annulus easily fits in our setting. If g is a homeomorphism of $S^1 \times [-1, 0]$ isotopic to the identity we can insert this dynamics in \mathbb{A} and trivially extend it to the whole annulus. If K is a g -invariant continuum that does not intersect $S^1 \times \{0\}$ and such that its complement is homeomorphic to $S^1 \times (-1, 0]$ then $X = K \cup S^1 \times (-\infty, -1]$ satisfies the same properties in \mathbb{A} .

Suppose now that h is an orientation-preserving homeomorphism of the open annulus $S^1 \times \mathbb{R}$ and X is a closed h -invariant set which is adherent to the lower end e^- but not to the upper end. Then, for large r we can take a global isotopy $\{I_t\}_{t=0}^1$, $I_0 = \text{id}$, which leaves X fixed and satisfies $I_1(h(S^1 \times \{r\})) = S^1 \times \{r\}$. In order to examine dynamical properties of X we can cut off $S^1 \times [r, +\infty)$ and work with the map $I_1 \circ h$, which is equal to h in a neighborhood of X .

Let K be a planar continuum invariant under an orientation-preserving planar homeomorphism f . Again, composing f with a suitable isotopy of \mathbb{R}^2 which is equal to the identity around K , we can assume f leaves a large disk that contains K invariant. The complement of K in that disk is homeomorphic to $S^1 \times (-1, 0]$. By Cartwright–Littlewood Theorem [CL51], K contains a fixed point p . We can puncture the disk at p and work with the induced dynamics in the resulting half-open annulus that leaves invariant the closed set $X = K \setminus \{p\}$.

Even though we already saw that some other typical settings adjust well to the half-open annulus, there is an important issue concerning \mathbb{A} which must be addressed: rotation of divergent orbits may not be well-defined. For example, consider the fibered rotation $h_\varphi: (\theta, r) \mapsto (\theta + \varphi(r), r)$ in \mathbb{A} . Let $p \in \mathbb{A}$ be a point whose forward orbit under a map f converges to the lower end. If φ is not bounded, the rotation number of p under f and the rotation number of $h_\alpha(p)$ under $h_\alpha \circ f \circ h_\alpha^{-1}$ may be different. A clear exposition of this phenomenon and how to navigate through it can be found in Le Roux [L13].

In our setting we will only consider orbits which stay away from the lower end of \mathbb{A} . Obviously, this is true for every orbit when we insert a dynamics of the closed annulus into \mathbb{A} . However, in general we require a compactness hypothesis on the orbits under consideration that will be read off the dynamics of F in $\tilde{\mathbb{A}}$. Denote by $\mathcal{O}^+(p)$, $\mathcal{O}^-(p)$ the forward and backward semi-orbit of a point $p \in \tilde{\mathbb{A}}$ under F .

Definition 3. A semi-orbit $\mathcal{O}^+(p)$ or $\mathcal{O}^-(p)$ is said to be *bounded* if it is contained in $\mathbb{R} \times [-r, 0]$ for some $r \leq 0$ or, equivalently, if its projection onto \mathbb{A} is relatively compact.

Our work concerns accessible points and prime ends of associated to X , so we describe X from its complement. The objects used for this purpose will be arcs in $\mathbb{A} \setminus X$. In this work, the term arc (resp. half-open arc) is used to refer to injective maps $\gamma: I \rightarrow \mathbb{A}$, where $I = [0, 1]$ (resp. $I = [0, 1)$), but mainly also to refer to their images $\gamma(I)$, which will be usually denoted γ as well.

3. THEORY OF PRIME ENDS

The theory of prime ends allows to replace the boundary of a simply connected domain U in the Riemann sphere (and such that $S^2 \setminus U$ contains two points at least) with an “ideal” boundary homeomorphic to a circle. Carathéodory proved that a conformal map $\phi: \mathbb{D} \rightarrow U$ extends to a homeomorphism between $\overline{\mathbb{D}}$ and \overline{U} if and only if ∂U is locally connected. As part of his research on conformal mappings,

he presented the theory of prime ends in [C13]. A brief account of the theory is presented here, for more information and proofs the reader is referred to [CL66, Ma82, Po91].

Let K be a continuum in \mathbb{R}^2 which consists of more than one point. A *cross-cut* c of $V = \mathbb{R}^2 \setminus K$ is an arc whose endpoints lie in K and is otherwise contained in V . The cross-cut c splits V into two connected components, one of which is bounded and denoted by $V(c)$. A sequence $\{c_n\}_{n \geq 1}$ of pairwise disjoint cross-cuts such that $V(c_{n+1}) \subset V(c_n)$ is called a *chain* (of cross-cuts). A chain $\{c'_n\}_n$ is said to divide another chain $\{c_n\}_n$ if for every i there exists j such that $V(c'_j) \subset V(c_i)$. Two chains are *equivalent* if any of them divides the other. A *prime end* is a minimal equivalence class in the set of chains in the sense that if $\{c_n\}_n$ is a representative of a class and $\{c'_n\}_n$ another chain which divides $\{c_n\}_n$ then they are equivalent, i.e., $\{c'_n\}_n$ also belongs to the class. The set of prime ends of V is denoted $\mathbb{P}(V)$.

Given a cross-cut c of V , denote $\widehat{V}(c)$ the union of $V(c)$ and the set of prime ends represented by chains $\{c_n\}_n$ such that $V(c_n) \subset V(c)$ for every n . The disjoint union $V \sqcup \mathbb{P}(V)$ is given a topology whose basis are open subsets of V and sets of the form $\widehat{V}(c)$ for some cross-cut c of V . The main theorem in the theory states that the topological space $\widehat{V} = V \sqcup \mathbb{P}(V)$, called the *prime end compactification* of V , is homeomorphic to $\overline{\mathbb{D}}$.

The *principal set* $\Pi(\mathfrak{p})$ of a prime end \mathfrak{p} consists of the points which are the limit of a sequence of cross-cuts in a chain that represents \mathfrak{p} . The principal set is compact and connected. As stated in the introduction, a point $p \in \partial K = \partial V$ is said to be *accessible* (from V) if there exists an arc $\gamma: [0, 1] \rightarrow V \cup \{p\}$ such that $\gamma([0, 1)) \subset V$ and $\gamma(1) = p$. Accessible points are dense in ∂V . The pair (p, γ) determines a way to approach the boundary of V and, consequently, determines a unique prime end \mathfrak{p} that satisfies $\Pi(\mathfrak{p}) = \{p\}$. Conversely, if $\Pi(\mathfrak{p}) = \{p\}$ then p is accessible from V . Prime ends for which their principal set is a singleton will be called accessible. In the literature they are often called prime ends of first or second kind (depending on whether their impression $\cap_n \widehat{V}(c_n)$ is a singleton or not for any chain $\{c_n\}_n$ that represents it). Accessible prime ends form a dense set in $\mathbb{P}(V)$. Notice that an accessible point in the boundary of V may determine several different accessible prime ends.

A half-open arc $\gamma: [0, 1) \rightarrow V$ is said to *determine* a prime end \mathfrak{p} if \mathfrak{p} is the limit of $\gamma(t)$ as $t \rightarrow 1$ if γ is viewed as an arc in \widehat{V} . Equivalently, γ determines \mathfrak{p} iff $\gamma^{-1}(c_n)$ is compact and non-empty if n is large enough for every chain $\{c_n\}_n$ that represents \mathfrak{p} . If \mathfrak{p} is accessible and p is its principal point then any half-open arc that determines \mathfrak{p} can be extended to a closed arc by $\gamma(1) = p$. Clearly, for any two different prime ends we can find disjoint arcs that determine them.

3.1. Line of prime ends. Suppose now that X is a closed non-empty subset of \mathbb{A} such that $U = \mathbb{A} \setminus X$ is homeomorphic to \mathbb{A} hence to $S^1 \times (-1, 0]$. Note that if we collapse $\partial \mathbb{A}$ to a point we are left with a simply connected planar domain. The theory of prime ends compactifies U with a boundary circle, the set $\mathbb{P}(U)$ of prime ends of U , and produces a set homeomorphic to a closed annulus $S^1 \times [-1, 0]$. $\mathbb{P}(U)$ is identified with $S^1 \times \{-1\}$.

Recall that $\tilde{\mathbb{A}} = \mathbb{R} \times (-\infty, 0]$, $\pi: \tilde{\mathbb{A}} \rightarrow \mathbb{A}$ is the universal cover and set $\tilde{X} = \pi^{-1}(X)$. Define a cross-cut \tilde{c} of \tilde{U} as the lift to $\tilde{\mathbb{A}}$ of a cross-cut c of U that does not intersect $\partial \mathbb{A}$. Then, \tilde{c} separates $\tilde{U} = \tilde{\mathbb{A}} \setminus \tilde{X}$ in two components, only one of which does not contain $\partial \mathbb{A}$, say $\tilde{V}(\tilde{c})$, and $\pi^{-1}(V(c)) = \cup_{\pi(\tilde{c})=c} \tilde{V}(\tilde{c})$. Then, we define the set $\mathbb{P}(\tilde{U})$ of prime ends of \tilde{U} as the set of minimal equivalence classes of chains of cross-cuts. The “compactification” $\tilde{U} \sqcup \mathbb{P}(\tilde{U})$ is given a topology through a basis composed of all open sets of \tilde{U} and all sets of the form $\widehat{\tilde{V}}(\tilde{c})$. Since π restricted to $\tilde{V}(\tilde{c})$ is injective, it induces a covering map $\hat{\pi}: \mathbb{P}(\tilde{U}) \rightarrow \mathbb{P}(U)$. Moreover, the generator of the deck transformations $T: \mathbb{A} \rightarrow \mathbb{A}$ given by $T(\theta, r) = (\theta + 1, r)$ also induces a generator of the deck transformations $\hat{T}: \mathbb{P}(\tilde{U}) \rightarrow \mathbb{P}(\tilde{U})$. Therefore, $\hat{\pi}: \mathbb{P}(\tilde{U}) \rightarrow \mathbb{P}(U)$ is a universal cover and we call $\mathbb{P}(\tilde{U})$ *line of prime ends* of \tilde{U} . The set $\widehat{\tilde{U}} = \tilde{U} \cup \mathbb{P}(\tilde{U})$ will be called *prime end closure* of \tilde{U} and it can be identified to $\mathbb{R} \times [-1, 0]$, where $\mathbb{P}(\tilde{U})$ corresponds to $\mathbb{R} \times \{-1\}$ and \hat{T} to the translation by $+1$.

The notions of accessible prime end and arc that determines a prime end translate verbatim to $\mathbb{P}(\tilde{U})$. In order to illustrate the idea of order, which is already present in $\mathbb{P}(\tilde{U})$, we introduce the following definition:

Definition 4. An arc $\gamma: [0, 1] \rightarrow \tilde{\mathbb{A}}$ (resp. a half-open arc $\gamma: [0, 1) \rightarrow \tilde{\mathbb{A}}$) is called a *hanging arc* (resp. *open hanging arc*) provided $\gamma(t) \in \partial \tilde{\mathbb{A}} = \mathbb{R} \times \{0\}$ iff $t = 0$. When it is defined, $\gamma(1)$ is said to be the *landing point* of γ .

Note that an open hanging arc in \tilde{U} that determines a prime end in $\mathbb{P}(\tilde{U})$ separates $\tilde{\mathbb{A}} \setminus \tilde{X}$ in exactly two connected components. The usual order in \mathbb{R} induces, by the identification of $\mathbb{P}(\tilde{U})$ to $\mathbb{R} \times \{-1\}$, a total order relation in $\mathbb{P}(\tilde{U})$ denoted \prec . This order relation is easily illustrated as follows: given $\mathfrak{p} \neq \mathfrak{q} \in \mathbb{P}(\tilde{U})$, $\mathfrak{p} \prec \mathfrak{q}$ if and only if there are disjoint open hanging arcs $\gamma_{\mathfrak{p}}, \gamma_{\mathfrak{q}}$ contained in $\tilde{\mathbb{A}} \setminus \tilde{X}$ that determine $\mathfrak{p}, \mathfrak{q}$, respectively, and such that $(\gamma_{\mathfrak{p}}(0))_1 < (\gamma_{\mathfrak{q}}(0))_1$ or, in words, $\gamma_{\mathfrak{p}}$ lies on the left of $\gamma_{\mathfrak{q}}$.

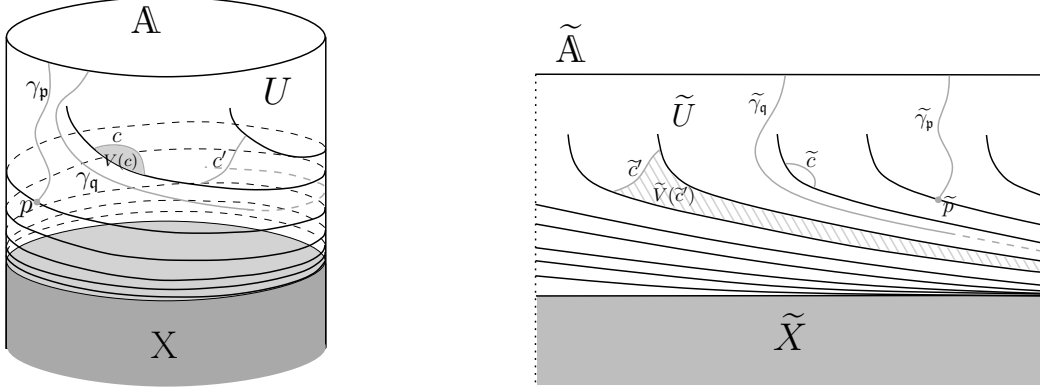


FIGURE 1. X is composed of two strings wrapped around \mathbb{A} that accumulate in a limit circle and the cylinder left below (in dark gray). $\gamma_{\mathfrak{p}}$ is a hanging arc that lands at p , $\gamma_{\mathfrak{q}}$ is an open hanging arc that determines a prime end that is not accessible. Cross-cuts \tilde{c}, \tilde{c}' are lifts of c, c' . Note that $\tilde{V}(\tilde{c})$ is compact, whereas $\tilde{V}(\tilde{c}')$ is not.

3.2. Prime end rotation number. Let $f: \mathbb{A} \rightarrow \mathbb{A}$ be an orientation-preserving homeomorphism and fix a lift F of f to $\tilde{\mathbb{A}}$. Assume X is invariant under f , hence so is U . The map f induces a homeomorphism $\hat{f}: \hat{U} \rightarrow \hat{U}$. There is a unique lift \hat{F} of \hat{f} to \tilde{U} which coincides with F in \tilde{U} .

The prime end rotation number is defined as

$$\hat{\rho}(F, X) = \lim_{n \rightarrow \infty} \frac{(\hat{F}^n(\mathfrak{p}))_1}{n} \in \mathbb{R},$$

where \mathfrak{p} is any prime end in $\mathbb{P}(\tilde{U})$ and we use the identification of $\mathbb{P}(\tilde{U})$ to $\mathbb{R} \times \{-1\}$ to compute $(\hat{F}^n(\mathfrak{p}))_1$. Note that $\hat{\rho}(F, X)$ only depends on the choice of lift F of f up to an integer constant.

Since the speed of rotation of every orbit of a circle homeomorphism is equal, we can describe $\hat{\rho}(F, X)$ by looking at the orbit of one prime end of $\mathbb{P}(\tilde{U})$, say \mathfrak{p} . Take \mathfrak{q} be another prime end as a reference. For any $n \in \mathbb{Z}$, there exists $m = m(n)$ such that $\hat{T}^m(\mathfrak{q}) \preceq \hat{F}^n(\mathfrak{p}) \prec \hat{T}^{m+1}(\mathfrak{q})$. Clearly, $\hat{\rho}(F, X) = \lim_n \frac{m(n)}{n}$.

Let β be a half-open arc in $\tilde{\mathbb{A}} \setminus \tilde{X}$ which determines the prime end \mathfrak{q} , $\beta \cap T\beta = \emptyset$, and denote by D the domain in $\tilde{\mathbb{A}} \setminus \tilde{X}$ enclosed by β and $T\beta$. The preceding inequality implies that there is an open hanging arc γ_n contained in $T^m D$ which determines $\hat{F}^n(\mathfrak{p})$.

4. RELATIVE WINDING NUMBER

Let γ, γ' be two hanging arcs such that $(\gamma(0))_1 < (\gamma'(0))_1$ and $\gamma(t) \neq \gamma'(t)$ for every $t \in [0, 1]$. Consider $\phi: [0, 1] \rightarrow S^1$ defined by

$$\phi(t) = \frac{\gamma(t) - \gamma'(t)}{\|\gamma(t) - \gamma'(t)\|}.$$

Notice that $\phi(0) = 1$. Let $\mathbb{R} \rightarrow S^1 : x \mapsto e^{2\pi i x}$ be the universal covering projection. The map ϕ has a unique lift $\tilde{\phi}: [0, 1] \rightarrow \mathbb{R}$ such that $\tilde{\phi}(0) = 0$.

Definition 5. The *relative winding number* of the hanging arcs γ and γ' is defined by

$$w(\gamma, \gamma') := \tilde{\phi}(1) \in \mathbb{R}.$$

Remark 6. We may replace ϕ in the definition with any map ϕ' homotopic to ϕ relative to $\{0, 1\}$. Indeed, the unique lift $\tilde{\phi}'$ of ϕ' such that $\tilde{\phi}'(0) = 0$ satisfies $\tilde{\phi}'(1) = \tilde{\phi}(1)$. In particular, if $\varphi_s: [0, 1] \rightarrow [0, 1]$ is a continuous family of reparameterizations of γ , $\varphi_0 = \text{id}$, it follows that $w(\gamma, \gamma') = w(\gamma \circ \varphi_1, \gamma')$ as long as

the relative winding number is well-defined for every pair $(\gamma \circ \varphi_s, \gamma')$, that is $\gamma(\varphi_s(t)) \neq \gamma'(t)$ for every $s, t \in [0, 1]$.

Given $a \in \mathbb{R}$, $a \notin 1/2 + \mathbb{Z}$, denote $[a]$ the closest integer to a . The following lemma shows that if we move the hanging arcs γ, γ' their relative winding number does not oscillate significantly as long as their landing points do not exchange sides.

Lemma 7. *Let $\{\gamma_s\}_{s=0}^1, \{\gamma'_s\}_{s=0}^1$ be two homotopies of hanging arcs such that $(\gamma_s(0))_1 < (\gamma'_s(0))_1$, $\gamma_s(t) \neq \gamma'_s(t)$ and*

$$(\gamma_s(1))_1 \neq (\gamma'_s(1))_1$$

for all $s, t \in [0, 1]$. Then,

$$[w(\gamma_0, \gamma'_0)] = [w(\gamma_1, \gamma'_1)].$$

Furthermore, if the homotopies of hanging arcs fix the landing points, then

$$w(\gamma_0, \gamma'_0) = w(\gamma_1, \gamma'_1).$$

Proof. The condition in the statement implies

$$\phi_s(1) = \frac{\gamma'_s(1) - \gamma_s(1)}{\|\gamma'_s(1) - \gamma_s(1)\|} \neq e^{i\pi/2}, e^{-i\pi/2},$$

hence $\widetilde{\phi}_s(1) = w(\gamma_s, \gamma'_s) \notin 1/2 + \mathbb{Z}$, for every $s \in [0, 1]$. The result trivially follows. \square

To sum up, if the landing points remain fixed by the homotopies the relative winding number is constant as long as it is defined.

Lemma 8. *Let γ, γ' be hanging arcs such that $(\gamma(0))_1 < (\gamma'(0))_1$.*

- *If $(\gamma'(1))_1 < \min(\gamma)_1$ then $[w(\gamma, \gamma')] = -1$.*
- *If $\max(\gamma')_1 < (\gamma(1))_1$ then $[w(\gamma, \gamma')] = 1$.*

Proof. For the first statement, define $\phi_r: [0, 1] \rightarrow S^1$ as follows

$$\phi_r(t) = \begin{cases} \frac{\gamma(0) - \gamma'(2t)}{\|\gamma(0) - \gamma'(2t)\|} & 0 \leq t \leq 1/2 \\ \frac{\gamma(2t-1) - \gamma'(1)}{\|\gamma(2t-1) - \gamma'(1)\|} & 1/2 \leq t \leq 1. \end{cases}$$

It is easy to check, see Remark 6, that ϕ_r is homotopic to ϕ relative to $\{0, 1\}$. Thus, its lift $\widetilde{\phi}_r$ satisfies $\widetilde{\phi}_r(1) = \widetilde{\phi}(1) = w(\gamma, \gamma')$. From the definition it is easy to see that $\widetilde{\phi}_r(t) \in (-1, 0)$ for $0 < t \leq 1/2$. Since $(\gamma'(1))_1 < (\gamma(0))_1$, $\widetilde{\phi}_r(1/2) \in (-1, -1/2)$. Moreover, $(\gamma'(1))_1 < (\gamma(t))_1$ for every t , so $\widetilde{\phi}_r(t) \in (-1/2, -3/2)$ for all $1/2 \leq t \leq 1$. Then, $[w(\gamma, \gamma')] = [\widetilde{\phi}_r(1)] = -1$.

The second statement is proven in a similar fashion. Consider $\phi_l: [0, 1] \rightarrow S^1$ defined by

$$\phi_l(t) = \begin{cases} \frac{\gamma(2t) - \gamma'(0)}{\|\gamma(2t) - \gamma'(0)\|} & 0 \leq t \leq 1/2 \\ \frac{\gamma(1) - \gamma'(2t-1)}{\|\gamma(1) - \gamma'(2t-1)\|} & 1/2 \leq t \leq 1. \end{cases}$$

Again, ϕ_l is homotopic to ϕ relative to $\{0, 1\}$ and $\widetilde{\phi}_l(1) = \widetilde{\phi}(1) = w(\gamma, \gamma')$. A description of $\widetilde{\phi}_l$ allows to finish the proof. \square

4.1. Accessible prime ends. Since we will study the prime ends of $\widetilde{U} = \widetilde{\mathbb{A}} \setminus \widetilde{X}$, we need the hanging arcs to stay outside \widetilde{X} .

Definition 9. A hanging arc γ such that $\gamma[0, 1) \subset \widetilde{\mathbb{A}} \setminus \widetilde{X}$ is called a *lead line*.

The term *hair* was used in [H16] for this purpose, but we think *lead line* expresses better the idea of an arc used to thoroughly examine all hidden cavities and corners of $\partial\widetilde{X}$. An example of lead line is $\gamma_{\mathbf{p}}$ from Figure 1.

Given two different accessible prime ends $\mathbf{p}, \mathbf{q} \in \mathcal{P}(\widetilde{U})$ whose principal points are p, q , respectively, there always exist disjoint lead lines $\gamma_{\mathbf{p}}, \gamma_{\mathbf{q}}$ such that $\gamma_{\mathbf{p}}(1) = p, \gamma_{\mathbf{q}}(1) = q$. Note that any two lead lines that determine \mathbf{p} are homotopic relative to the landing point p .

Definition 10. Let $\mathbf{p}, \mathbf{q} \in \mathbb{P}(\tilde{U})$ be different accessible prime ends such that $\mathbf{p} \prec \mathbf{q}$ and $\gamma_{\mathbf{p}}, \gamma_{\mathbf{q}}$ as above. The relative winding number of \mathbf{p}, \mathbf{q} is defined by

$$w(\mathbf{p}, \mathbf{q}) = w(\gamma_{\mathbf{p}}, \gamma_{\mathbf{q}}).$$

Proposition 11. $w(\mathbf{p}, \mathbf{q})$ does not depend on the choice of $\gamma_{\mathbf{p}}, \gamma_{\mathbf{q}}$.

Proof. Let $\gamma'_{\mathbf{p}}, \gamma'_{\mathbf{q}}$ be another pair of lead lines satisfying the properties above. We have to show that $w(\gamma_{\mathbf{p}}, \gamma_{\mathbf{q}}) = w(\gamma'_{\mathbf{p}}, \gamma'_{\mathbf{q}})$. Note that $(\gamma_{\mathbf{p}}(0))_1 < (\gamma_{\mathbf{q}}(0))_1, (\gamma'_{\mathbf{p}}(0))_1 < (\gamma'_{\mathbf{q}}(0))_1$.

Assume first that $(\gamma_{\mathbf{p}}(0))_1 < (\gamma'_{\mathbf{q}}(0))_1$. Let $\{\gamma_{\mathbf{p}}^s\}_{s=0}^1$ be a homotopy of lead lines between $\gamma_{\mathbf{p}}^0 = \gamma'_{\mathbf{p}}$ and $\gamma_{\mathbf{p}}^1 = \gamma_{\mathbf{p}}$ that fixes the landing point p and avoids $\gamma'_{\mathbf{q}}(0)$. Similarly, let $\{\gamma_{\mathbf{q}}^s\}_{s=0}^1$ be a homotopy of lead lines between $\gamma_{\mathbf{q}}^0 = \gamma'_{\mathbf{q}}$ and $\gamma_{\mathbf{q}}^1 = \gamma_{\mathbf{q}}$ such that $\gamma_{\mathbf{q}}^s(1) = q$ and $\gamma_{\mathbf{q}}^s(0) \neq \gamma_{\mathbf{p}}(0)$ for every s . Since the arcs $\gamma_{\mathbf{p}}^s$ are disjoint to q , if $\epsilon > 0$ is small enough we have that $\gamma_{\mathbf{p}}^s[0, 1] \cap \gamma'_{\mathbf{q}}[1 - \epsilon, \epsilon] = \emptyset$ and also that $\gamma_{\mathbf{p}}^s[0, \epsilon] \cap \gamma'_{\mathbf{q}}[0, 1] = \emptyset$. Likewise, we may fix $\epsilon > 0$ that additionally satisfies

$$\gamma_{\mathbf{q}}^s[0, \epsilon'] \cap \gamma_{\mathbf{p}}[0, 1] = \emptyset \quad \text{and} \quad \gamma_{\mathbf{q}}^s[0, 1] \cap \gamma_{\mathbf{p}}[1 - \epsilon', \epsilon'] = \emptyset$$

The idea is to use ϵ to construct a reparameterization φ of $[0, 1]$ so that $w(\gamma_{\mathbf{p}}^s, \gamma'_{\mathbf{q}} \circ \varphi)$ and $w(\gamma_{\mathbf{p}}, \gamma_{\mathbf{q}}^s \circ \varphi)$ is well-defined for every s . It suffices that $\varphi[0, \epsilon] \supset [0, 1 - \epsilon]$. Then,

$$w(\gamma'_{\mathbf{p}}, \gamma'_{\mathbf{q}}) = w(\gamma'_{\mathbf{p}}, \gamma'_{\mathbf{q}} \circ \varphi) = w(\gamma_{\mathbf{p}}, \gamma'_{\mathbf{q}} \circ \varphi) = w(\gamma_{\mathbf{p}}, \gamma_{\mathbf{q}} \circ \varphi) = w(\gamma_{\mathbf{p}}, \gamma_{\mathbf{q}}),$$

where the first and last equality comes from Remark 6 and the other two from Lemma 7.

In order to conclude the proof, notice that if the hypothesis $(\gamma_{\mathbf{p}}(0))_1 < (\gamma'_{\mathbf{q}}(0))_1$ does not hold then $(\gamma'_{\mathbf{p}}(0))_1 < (\gamma_{\mathbf{q}}(0))_1$ and the procedure can be reversed: first use the homotopy between $\gamma'_{\mathbf{q}}$ and $\gamma_{\mathbf{q}}$ and then a suitable reparameterization of the homotopy between $\gamma'_{\mathbf{p}}$ and $\gamma_{\mathbf{p}}$. \square

Our immediate goal is to translate Lemma 8 to the language of accessible prime ends.

Definition 12. For an accessible prime end $\mathbf{p} \in \mathbb{P}(\tilde{U})$, set

$$\min \mathbf{p} = \sup_{\gamma \in \Gamma(\mathbf{p})} \min(\gamma)_1, \quad \max \mathbf{p} = \inf_{\gamma \in \Gamma(\mathbf{p})} \max(\gamma)_1,$$

where $\Gamma(\mathbf{p})$ denotes the set of lead lines γ which determine the prime end \mathbf{p} .

Clearly, any lead line γ which determines \mathbf{p} satisfies $[\min \mathbf{p}, \max \mathbf{p}] \subset (\gamma)_1$. Furthermore, it can be shown that given $\epsilon > 0$ it is possible to construct γ_{ϵ} such that $(\gamma_{\epsilon})_1 \subset [\min \mathbf{p} - \epsilon, \max \mathbf{p} + \epsilon]$.

A straightforward corollary of Lemma 8 reads as follows.

Lemma 13. Let $\mathbf{p}, \mathbf{q} \in \mathbb{P}(\tilde{U})$ be accessible prime ends, $\mathbf{p} \prec \mathbf{q}$, and p, q be their principal points, respectively.

- If $(q)_1 < \min \mathbf{p}$ then $[w(\mathbf{p}, \mathbf{q})] = -1$.
- If $\max \mathbf{q} < (p)_1$ then $[w(\mathbf{p}, \mathbf{q})] = 1$.

5. TRANSVERSE DRIFT

Recall that a semi-orbit $\mathcal{O}^+(p)$ or $\mathcal{O}^-(p)$ in $\tilde{\mathbb{A}}$ is bounded if its projection to \mathbb{A} is contained in a compact subset of \mathbb{A} .

Theorem 14. There do not exist accessible points $p, q \in \tilde{X}$ such that $\mathcal{O}^-(p)$ and $\mathcal{O}^+(q)$ are bounded that

$$\lim_{n \rightarrow -\infty} (F^n(p))_1 - n\rho = -\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} (F^n(q))_1 - n\rho = +\infty,$$

where $\rho = \hat{\rho}(F, X)$.

Notice that this result is more general than Theorem 2.(1) as it covers the closed annulus case.

The homotopy between f and the identity map lifts to a homotopy $\{h_s\}_{s \in [0, 1]}$. It follows that the displacement of a point x under the homotopy can be bounded

$$|(x)_1 - (h_s(x))_1| < D(x), \quad \forall s \in [0, 1],$$

by a map $D: \tilde{\mathbb{A}} \rightarrow \mathbb{R}^+$ which only depends on the second coordinate of $x \in \tilde{\mathbb{A}} = \mathbb{R} \times (-\infty, 0]$.

As an application of Lemma 7 we have:

Proposition 15. Let $\mathfrak{c}, \mathfrak{c}'$ be accessible prime ends in \tilde{X} and e, e' their principal points, respectively. If $|(e)_1 - (e')_1| > D(e) + D(e')$ then

$$[w(\mathfrak{c}, \mathfrak{c}')] = [w(\hat{F}(\mathfrak{c}), \hat{F}(\mathfrak{c}'))].$$

Proof of Theorem 14. Choose \mathfrak{c} an accessible prime end of $\tilde{\mathbb{A}} \setminus \tilde{X}$ as a point of reference. Denote $\mathfrak{p}_n, \mathfrak{q}_n$ the prime ends obtained from $\hat{F}^n(\mathfrak{p}), \hat{F}^n(\mathfrak{q})$ by translation under \hat{T} so that $\hat{T}^{-1}\mathfrak{c} \prec \mathfrak{q}_n \prec \mathfrak{c} \prec \mathfrak{p}_n \prec \hat{T}\mathfrak{c}$, and p_n, q_n their principal points, respectively. Assume without loss of generality that $\mathfrak{p}_0 = \mathfrak{p}, \mathfrak{q}_0 = \mathfrak{q}$.

Claim: $\limsup_{n \rightarrow -\infty} (p_n)_1 = -\infty$, $\lim_{n \rightarrow +\infty} (q_n)_1 = +\infty$.

Indeed, if $\mathfrak{q}_n = \hat{T}^{-m(n)}\hat{F}^n(\mathfrak{q})$ then the numbers $m(n) - n\rho$ are bounded by a constant which is independent of n . Since $(q_n)_1 + (m(n) - n\rho) = (F^n(q))_1 - n\rho$, it follows that $\lim_{n \rightarrow +\infty} (q_n)_1 = +\infty$. We can argue similarly for p_n .

Set $C = \max\{D(x) : x \in \mathcal{O}^-(p) \cup \mathcal{O}^+(q)\} = \max\{D(x) : x \in \{p_n\}_{n \leq 0} \cup \{q_n\}_{n \geq 0}\}$. It is finite because both semi-orbits are bounded.

Choose first k_0 so that $(q_k)_1 > C + 1$ for every $k \geq k_0$. By Lemma 13, if $(p_n)_1 < \min \mathfrak{q}_{k_0}$ then $[w(\mathfrak{q}_{k_0}, \mathfrak{p}_n)] = -1$. Take n_0 such that for every $n \leq n_0$ the previous condition on the relative winding number is satisfied as well as $(p_n)_1 < -C - 1$. If we now pick k_1 so that $(q_{k_1})_1 > 1 + \max \mathfrak{p}_{n_0}$ we deduce also from Lemma 13 that $[w(\mathfrak{q}_{k_1}, \hat{T}^j \mathfrak{p}_{n_0})] = 1$ for every $j \in \{-1, 0, 1\}$. The configuration is sketched in Figure 2.

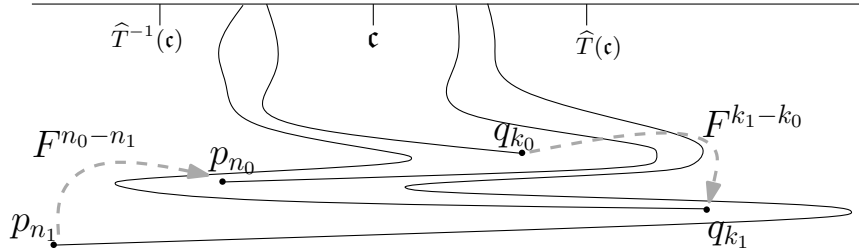


FIGURE 2.

To conclude the proof, set $n_1 = n_0 - (k_1 - k_0)$. For any $0 \leq i \leq n_0 - n_1 = k_1 - k_0$, if $q_{k_0+i} = T^m F^i(q_{k_0})$ and $p_{n_1+i} = T^{m'} F^i(p_{n_1})$ then $|m - m'| \leq 1$. As a consequence,

$$(F^i(q_{k_0}))_1 - (F^i(p_{n_1}))_1 \geq (q_{k_0+i})_1 - (p_{n_1+i})_1 + 1 > 2C \geq D(F^i(q_{k_0})) + D(F^i(p_{n_1}))$$

and we can apply Proposition 15 to obtain

$$[w(\hat{F}^i(\mathfrak{q}_{k_0}), \hat{F}^i(\mathfrak{p}_{n_1}))] = [w(\hat{F}^{i+1}(\mathfrak{q}_{k_0}), \hat{F}^{i+1}(\mathfrak{p}_{n_1}))].$$

Thus, $[w(\mathfrak{q}_{k_0}, \mathfrak{p}_{n_1})] = [w(\hat{F}^{k_1-k_0} \mathfrak{q}_{k_0}, \hat{F}^{n_0-n_1} \mathfrak{p}_{n_1})] = [w(\mathfrak{q}_{k_1}, \hat{T}^j(\mathfrak{p}_{n_0}))]$ for some $j \in \{-1, 0, 1\}$, which is absurd. \square

Remark 16. The speed of convergence in the limits of Theorem 14 does not play a role.

Corollary 17. If the full orbit of a point $z \in X$ has well-defined rotation number, that is, $\rho_z = \lim_{n \rightarrow \pm\infty} (F^n(\tilde{z}))_1 / n \in \mathbb{R}$ for a lift \tilde{z} of z and $\rho_z \neq \hat{\rho}(F, X)$ then z is not an accessible point of X .

Theorem 18. Theorem 14 is not true if we replace \lim by \limsup in the statement.

Proof. We construct a counterexample for $\hat{\rho}(F, X) = 0$. It can be adapted to an arbitrary rotation number without much effort.

Fix $\theta_0 \neq \theta_1 \in S^1$ and consider a homeomorphism $h: \mathbb{A} \rightarrow \mathbb{A}$ given by $h(\theta, r) = (\theta, \varphi(\theta, r))$ and such that $\varphi(\theta_0, r) < r < \varphi(\theta_1, r)$ for every $r < 0$. The forward (resp. backward) semi-orbit of $x_0 = (\theta_0, -1)$ (resp. $x_1 = (\theta_1, -1)$) under h tends to e^- , the lower end of \mathbb{A} . We need the motion under h to be slow close to e^- , it suffices to ask for $|\varphi(r, \theta) - r| \rightarrow 0$ as $r \rightarrow -\infty$.

Consider $\phi: \mathbb{A} \rightarrow \mathbb{A}$ given by the change of coordinate $\theta \mapsto \theta + r \sin(2\pi r)$. Denote $g = \phi \circ h \circ \phi^{-1}$ and let $L(\theta) = \{(\theta, r) : r \leq 0\}$ the vertical line in \mathbb{A} . Choose $G: \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ a lift of g that fixes pointwise $\partial \tilde{\mathbb{A}}$. The lift of $\phi(L(\theta_i))$, $i = 0, 1$, to the universal cover $\tilde{\mathbb{A}}$ is a line that oscillates arbitrarily in the lift of the angular coordinate and so does the semi-orbit of a lift of $\phi(x_i)$ under G . By the properties of h , if $\iota_r: S^1 \times \{r\} \rightarrow \mathbb{A}$ is the inclusion map, $\|g|_{S^1 \times \{r\}} - \iota_r\| \rightarrow 0$ as $r \rightarrow -\infty$.

Now, we plug $\psi: \mathbb{A} \rightarrow S^1 \times (-2, -1]$ using the change of coordinates $r \mapsto -1 + r/(r^2 + 1)$. The convergence of $g|_{S^1 \times \{r\}}$ guarantees that we can extend the conjugate dynamics $\psi^{-1} \circ g \circ \psi$ to a homeomorphism $f: \mathbb{A} \rightarrow \mathbb{A}$ which is the identity in $S^1 \times ((-\infty, -2] \cup [-1, 0])$. The set $X = (\psi \circ \phi)(L(\theta_0) \cup L(\theta_1)) \cup S^1 \times (-\infty, -2]$ is closed and invariant under f . Let F be the lift of f that is equal to $\psi \circ G \circ \psi$ in $S^1 \times [-2, -1]$. The forward (resp. backward) semi-orbit of a lift q of $\psi \circ \phi(x_0)$ (resp. a lift p of $\psi \circ \phi(x_1)$) under F satisfies

$$\begin{aligned} \liminf_{n \rightarrow +\infty} (F^n(q))_1 &= -\infty \text{ and } \limsup_{n \rightarrow +\infty} (F^n(q))_1 = +\infty. \\ (\text{resp. } \liminf_{n \rightarrow -\infty} (F^n(p))_1 &= -\infty \text{ and } \limsup_{n \rightarrow -\infty} (F^n(p))_1 = +\infty). \end{aligned}$$

Since F fixes pointwise $\mathbb{R} \times \{-1\}$, F fixes every cross-cut contained in it and it follows that $\widehat{\rho}(F, X) = 0$. \square

5.1. Invariant measures. Every Borel probability measure on \mathbb{A} which is invariant under the action of f has an associated rotation number:

$$\rho(F, \mu) = \int_{\mathbb{A}} u \, d\mu,$$

where $u: \mathbb{A} \rightarrow \mathbb{R}$ is the displacement function defined by $u(z) = (F(\tilde{z}))_1 - (\tilde{z})_1$, where \tilde{z} is any lift of z .

Invariant measures can always be found in compact ω -limits of orbits (hence in compact invariant sets), for example as weak limits of measures of the form $\sum_{k=0}^n \delta_{f^k(x)}/n$.

Theorem 19. *Let μ be a Borel probability measure invariant under f and supported on X . If $\rho(F, \mu) \neq \widehat{\rho}(F, X)$ then the set of accessible points of X has zero μ -measure.*

Proof. Applying Birkhoff's Ergodic Theorem to F and F^{-1} ,

$$\rho(F, \mu) = \int_{\mathbb{A}} u \, d\mu = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} u \circ F^k(x) = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} u \circ F^{-k}(x)$$

for μ -almost every x . The limits on the right are equal to $\lim_{n \rightarrow \pm\infty} (F^n(x))_1/n$. This implies that the full orbit of μ -almost every point in X has a rotation number equal to $\rho(F, \mu)$. However, recall Corollary 17, an accessible point cannot have a well-defined rotation number different from $\widehat{\rho}(F, X)$. \square

6. UNBOUNDED DRIFTS

Assume there are accessible points $p, q \in \tilde{X}$ whose backward and forward semi-orbits, respectively, are bounded and their rotations drift from the expected one. In other words, $\{(F^n(p))_1 - n\rho\}_{n \leq 0}$ and $\{(F^n(q))_1 - n\rho\}_{n \geq 0}$ are unbounded for $\rho = \widehat{\rho}(F, X)$. We already know from Theorem 18 that both sequences can have subsequences that diverge to ∞ (in any direction). Nevertheless, as was proved in the previous section, the sequences themselves cannot tend to $+\infty$ and to $-\infty$, respectively, or viceversa. We shall see now that this qualitative obstruction disappears if the sequences diverge towards the same end of \mathbb{R} .

Theorem 20. *Given any $\rho_0 \in \mathbb{R}$, there exists $f: \mathbb{A} \rightarrow \mathbb{A}$ and an f -invariant closed set X such that $\widehat{\rho}(F, X) = \rho_0$ for a lift F of f and there are p, q accessible points of \tilde{X} such that*

$$\begin{aligned} \lim_{n \rightarrow -\infty} (F^n(p))_1 - n\rho_0 &= +\infty \\ \lim_{n \rightarrow +\infty} (F^n(q))_1 - n\rho_0 &= +\infty \end{aligned}$$

Proof. The invariant set X in the example will consist of several strings which spiral towards a limit circle together with the annulus bounded by this circle, as in Figure 1. The dynamics in the set of strings and also in the limit circle will be a rotation of angle ρ_0 (if ρ_0 is rational) or a Denjoy homeomorphism of rotation number ρ_0 (if ρ_0 is irrational). A small push in the radial coordinate will force the orbits of points in the strings to spiral towards or outwards the circle. Being a bit more involved, we only give details of the irrational case. We follow the notation and ideas employed in Theorem 18.

Suppose $\rho_0 \notin \mathbb{Q}$ and consider a Denjoy homeomorphism $\tau: S^1 \rightarrow S^1$ whose rotation number is $\rho_0 + \mathbb{Z}$. Let $\{\theta_n = \tau^n(\theta_0)\}_{n \in \mathbb{Z}}$ be a wandering orbit of τ . Consider $h: \mathbb{A} \rightarrow \mathbb{A}$ defined by $h(\theta, r) = (\tau(\theta), \varphi(\theta, r))$, where φ satisfies $\varphi(\theta_n, r) = r + r/n(r^2 + 1)$ for every $n \in \mathbb{Z} \setminus \{0\}, r \leq 0$. Note that φ vanishes in every point (θ, r) if θ belongs to Λ , the minimal set of τ . Take $x_0 = (\theta_0, -1)$ and observe that both the forward and backward orbit of x_0 under h tend to e^- .

We now twist \mathbb{A} using $\phi: \mathbb{A} \rightarrow \mathbb{A}$ defined by $\theta \mapsto \theta - r$. The image of every $L(\theta) = \{(\theta, r) : r \leq 0\}$ by ϕ is wrapped around \mathbb{A} infinitely many times in the positive direction. In the universal cover $\tilde{\mathbb{A}}$, $\phi(L(\theta))$ is lifted to $L'(\tilde{\theta}) = \{(\tilde{\theta} - r, r) : r \leq 0\}$. Denote $g = \phi \circ h \circ \phi^{-1}$ and let $\tilde{\tau}: \mathbb{R} \rightarrow \mathbb{R}$ be the lift of τ with rotation number ρ_0 and G a lift of g which coincides with $\tilde{\tau}$ in $\mathbb{R} \times \{0\}$. Since $L'(\tilde{\theta})$ drifts away from a lift of $L(\theta)$,

$$(G^n(\tilde{y}_0))_1 - \tilde{\tau}^n(\tilde{\theta}_0) \xrightarrow{n \rightarrow \pm\infty} +\infty,$$

where \tilde{y}_0 is a lift of $\phi(x_0)$. Notice that $|\tilde{\tau}^n(\tilde{\theta}_0) - n\rho_0|$ is bounded by a small constant independent of n .

In the slices $S^1 \times \{r\}$, $\|g|_{S^1 \times \{r\}} - \iota_r \tau \iota_r^{-1}\| \rightarrow 0$ as $r \rightarrow -\infty$. Thus, we can conjugate g by ψ to a homeomorphism in $S^1 \times (-2, -1]$ and extend it by $(\theta, r) \mapsto (\tau(\theta), r)$ to a global homeomorphism f in \mathbb{A} . Then, $X = (\psi \circ \phi)(\cup_{\theta \in \Lambda \cup \{\theta_n\}_{n \in \mathbb{Z}}} L(\theta)) \cup S^1 \times (-\infty, -2]$ and $p = q = \psi \circ \phi(x_0)$ satisfy the conditions in the statement. Note the dynamics in the set of twisted lines has not been transformed throughout the construction, hence the lift F of f determined by G and ultimately by $\tilde{\tau}$ must satisfy $\hat{\rho}(F, X) = \rho(\tilde{\tau}) = \rho_0$. \square

The previous result shows that the speed of convergence in the limits, in spite of going unnoticed in the transverse drift case, does play a role in general. To ease the exposition, we suppose from now on that $\limsup_{n \rightarrow +\infty} (F^n(q))_1 - n\rho = +\infty$, where $\rho = \hat{\rho}(F, X)$. We introduce now two mild conditions which prevent the drifting forward semi-orbit of q to reduce the speed of its drift to zero if $\rho \notin \mathbb{Q}$ or to accumulate exclusively in a set of periodic points whose rotation number is exactly ρ provided it is rational.

- (H1) $\limsup_{n \rightarrow +\infty} (F^n(q))_1 - n\rho' = +\infty$ for some $\rho' = a/b \in \mathbb{Q}$ with $\rho \leq \rho'$ and
- (H2) there are strictly increasing sequences of integers $\{m_i\}, \{n_i\}$ and an integer k , $0 \leq k < b$, such that $T^{-m_i - an_i} F^{bn_i + k}(q)$ converges to a point $q_0 \in \tilde{X}$ that is not fixed under $T^{-a} F^b$.

Lemma 21.

- (1) If there exists $\delta > 0$ such that the inequality $(F^{b+n}(q))_1 > (F^n(q))_1 + a + \delta$ holds for infinitely many positive integers n then (H2) holds.
- (2) As a consequence, if (H1) is satisfied for some $\rho' > \rho$ or, equivalently, $L = \limsup_{n \rightarrow +\infty} (F^n(q))_1/n > \rho$ then (H1) and (H2) holds.

Proof. (1) Let q_0 be a limit point of sequences of the form $T^{-m_i - an_i} F^{bn_i + k}(q)$ with all $bn_i + k$ satisfying the inequality. Then, $(T^{-a} F^b(q_0))_1 = (F^b(q_0))_1 - a > (q_0)_1 + \delta$ so q_0 cannot be fixed by $T^{-a} F^b$.

(2) Take $\rho' = a/b \in (\rho, L) \cap \mathbb{Q}$. It is easy to see that (1) holds for every $\delta > 0$ such that $b\delta < L - \rho'$. \square

Remark 22. Suppose $\rho \in \mathbb{Q}$ and (H1) is only valid for $\rho' = \rho$. Hypothesis (H2) is not satisfied if and only if every limit point of the forward orbit of $\pi(q)$ under f in the annulus is periodic and has rotation number ρ .

Lemma 23. For any $a \in \mathbb{N}, k \in \mathbb{Z}$ and a fixed choice of signs, the condition

$$\limsup_{n \rightarrow \pm\infty} (F^{an+k}(x))_1 - an\rho = \pm\infty \quad \text{is equivalent to} \quad \limsup_{n \rightarrow \pm\infty} (F^n(x))_1 - n\rho = \pm\infty$$

provided the semi-orbit of x under consideration is bounded.

Proof. The result follows from the inequality:

$$|((F^{an+k}(x))_1 - an\rho) - ((F^m(x))_1 - m\rho)| < |k\rho| + a \max\{D(y) : y \in \mathcal{O}^\pm(x)\}$$

that holds for any m, n such that $a(n-1) + k \leq m \leq a(n+1) + k$. \square

Theorem 24. Let $f: \mathbb{A} \rightarrow \mathbb{A}$ be an orientation-preserving homeomorphism, X a non-empty invariant closed set such that $\mathbb{A} \setminus X$ is homeomorphic to \mathbb{A} and $F: \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ and \tilde{X} lifts of f and X to the universal cover of \mathbb{A} . Suppose there exists an accessible point $q \in \tilde{X}$ with bounded forward semi-orbit and such that

$$\limsup_{n \rightarrow +\infty} (F^n(q))_1 - n\rho = +\infty,$$

where $\rho = \hat{\rho}(F, X)$, and (H1) and (H2) are satisfied. Then, there is a constant $C > 0$ such that

$$|(F^n(p))_1 - (p)_1 - n\rho| < C$$

for any $n \leq 0$ and any accessible point $p \in \tilde{X}$ whose backward semi-orbit is bounded.

Proof. First, we show that it is enough to prove the result in the case $\rho' = 0$ (from (H1)) and $k = 0$ (from (H2)). Indeed, take $\rho' = a/b$ and $k \in \mathbb{Z}$ satisfying (H1)–(H2) and set $q' = F^k(q)$. Lemma 23 implies that $\limsup_{n \rightarrow +\infty} (F^n(q))_1 - n\rho' = +\infty$ is equivalent to $\limsup_{n \rightarrow +\infty} (G^n(q))_1 = +\infty$, where $G = T^{-a}F^b$, which is in turn equivalent to $\limsup_{n \rightarrow +\infty} (G^n(q'))_1 = +\infty$. Property (H2) provides q_0 , the limit of $T^{-m_i}G^{n_i}(q')$, that is not fixed by G . Finally, $\limsup_{n \rightarrow -\infty} (G^n(p))_1 - n(\rho - \rho') = +\infty$ if and only if $\limsup_{n \rightarrow -\infty} (F^n(p))_1 - n\rho = +\infty$, again by Lemma 23. In sum, after replacing F by $G = T^{-a}F^b$ and q by q' we can assume $\rho' = k = 0$. This implies we can assume $\rho \leq 0$ as well.

The ensuing statements enclose the core of the proof. First, we need a technical lemma and its corollaries.

Lemma 25. *Let D be a closed disk and γ be a lead line that determines $\mathfrak{e} \in \mathbb{P}(\tilde{U})$. Suppose $\alpha = \gamma|_{[t_0, t_1]}$, $t_1 \neq 1$, is a connected component of $\gamma \cap D$. $D \setminus \alpha$ is partitioned into two connected sets W_1, W_2 . Then,*

- *either $W_1 \cap \tilde{X} \neq \emptyset \neq W_2 \cap \tilde{X}$ and there is a cross-cut c of \tilde{U} contained in ∂D such that $\mathfrak{e} \in \hat{V}(c)$,*
- *or at least one of $W_1 \cap \tilde{X}, W_2 \cap \tilde{X}$ is empty. Suppose $W_1 \cap \tilde{X} = \emptyset$. Clearly, there exists a closed disk $D' \subset W_2$ such that $D' \cap \tilde{X} = D \cap \tilde{X}$. Furthermore, for any neighborhood W of $W_1 \cup \{\alpha\}$ in \tilde{U} there exists $\epsilon > 0$ and a global isotopy $\{h_s\}_{s=0}^1$, $h_0 = \text{id}$ supported in W such that*

$$(h_1 \circ \gamma)^{-1}(W_2) = (h_1 \circ \gamma)^{-1}(D) \subset \gamma^{-1}(D) \setminus [t_0 - \epsilon, t_1 + \epsilon].$$

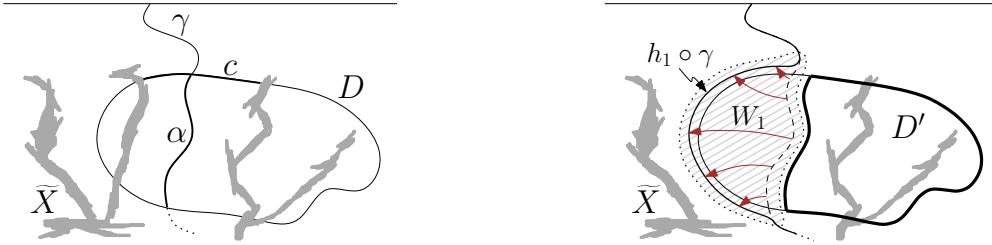


FIGURE 3. First alternative of Lemma 25 on the left. Second alternative on the right. The support of the isotopy is shadowed.

Outline of the proof. Firstly, notice that the result is trivial if α is a singleton. The characterization in the first alternative follows from the fact that \tilde{X} has no compact connected component, thus every component of \tilde{X} that meets W_i also meets ∂W_i . In the second case, we can construct the isotopy in two steps, first move α to ∂D within D and then push the resulting arc out of D . \square

Corollary 26. *Let D be a closed disk and γ be a lead line that determines an accessible prime end \mathfrak{e} .*

- (1) *Suppose there is a finite family $\{\alpha_1, \dots, \alpha_n\}$ of components of $\gamma \cap D$ such that the second alternative in the dichotomy of Lemma 25 holds. Then, we can find a closed disk $D' \subset D$ such that $D' \cap \alpha_i = \emptyset$ and $D' \cap \tilde{X} = D \cap \tilde{X}$.*
- (2) *Let γ^* be a hanging arc that does not meet D nor $\gamma(1) = \Pi(\mathfrak{e})$. There exists a closed disk $D' \subset D$ such that $D' \cap \tilde{X} = D \cap \tilde{X}$ and $\gamma[0, t^*] \cap D' = \emptyset$, where $t^* = \max\{t : \gamma(t) \in \gamma^*\}$.*
- (3) *Suppose the principal point of \mathfrak{e} does not belong to D . If γ does not cross any cross-cut of \tilde{U} contained in ∂D then there exists a lead line γ' disjoint to D that also determines \mathfrak{e} .*

Proof. (1) The arcs α_i are disjoint and divide D in $n + 1$ components. From the assumption we deduce that only one of these components intersects \tilde{X} and the conclusion follows.

(2) Denote \mathcal{C} the set of components of $\gamma[0, t^*] \cap D$. The arcs γ, γ^* split \mathbb{A} in several regions. Any of those regions whose boundary is entirely contained in $\gamma[0, t^*] \cup \gamma^* \cup \partial \mathbb{A}$ cannot contain any piece of \tilde{X} because \tilde{X} has no compact connected component. Thus, every $\alpha \in \mathcal{C}$ satisfies the second alternative in Lemma 25. We deduce that there is a single component W of $D \setminus (\cup_{\alpha \in \mathcal{C}} \alpha)$ that intersects \tilde{X} and we can construct a closed disk D' inside W such that $D' \cap \tilde{X} = D \cap \tilde{X}$.

(3) Let $\{\alpha_i\}_{i \in I}$ be the set of components of $\gamma \cap D$. After an arbitrary small perturbation of γ around ∂D supported in \tilde{U} , we can assume I is finite, $I = \{1, \dots, n\}$. By hypothesis, for any component α_i the second alternative in the lemma holds. Denote W_1^i the component of $D \setminus \alpha_i$ disjoint to \tilde{X} . Arrange the indices in order so that W_1^i is not contained in W_1^j if $j < i$. Then, using Lemma 25, there are global

isotopies h_s^i , $i = 1 \dots n$, which push α_i out of D and such that $\text{supp}(h_s^i)$ is disjoint to α_j for all $j > i$. The concatenation yields an isotopy h_s from the identity to h_1 that satisfies $(h_1 \circ \gamma)^{-1}(D) = \emptyset$. Moreover, the support of h_1 lies in \tilde{U} , hence $h_1 \circ \gamma$ is a lead line that determines \mathfrak{c} but misses D , as desired. \square

The following proposition is the first step towards the proof. Notice beforehand that the closed ball B will be defined as a small neighborhood of the limit point provided by (H2).

Proposition 27. *Suppose $\rho \leq 0$. Let B a closed ball which contains a sequence of the form $\{T^{-m_i}F^{n_i}(q)\}_{i=1}^\infty$ with $\{m_i\}_i, \{n_i\}_i$ increasing integer sequences and $q \in \tilde{X}$ an accessible point. In addition, B is free, i.e., $B \cap F(B) = \emptyset$. Then, we can find a prime end $\mathfrak{c} \in \mathcal{P}(\tilde{U})$ such that if \mathfrak{c} is an accessible prime end, $\mathfrak{c} \prec \mathfrak{c}$ and its principal point belongs to B then every lead line that determines $\hat{F}(\mathfrak{c})$ must go through B .*

Proof. Suppose the conclusion is false, that is, there is a sequence $\{\mathfrak{c}_k\}_{k \geq 1}$ of accessible prime ends such that $\mathfrak{c}_k \rightarrow -\infty$ in the line of prime ends of \tilde{U} and whose principal point lies in B and γ_k lead lines determining \mathfrak{c}_k such that $F(\gamma_k) \cap B = \emptyset$. Without loss of generality we can assume that the lead lines γ_k are pairwise disjoint.

For any prime end \mathfrak{c} define $\mathcal{A}^\mathfrak{c}$ as the set of accessible prime ends \mathfrak{q} , $\mathfrak{q} \prec \mathfrak{c}$, which are determined by lead lines that do not meet B .

Claim: $\mathcal{A}^\mathfrak{c}$ is positively invariant under \hat{F} for some $\mathfrak{c} \in \mathcal{P}(\tilde{U})$.

Take $\mathfrak{c} \in \mathcal{A}^\mathfrak{c}$ and a lead line $\gamma_\mathfrak{c}$ that determines \mathfrak{c} and is disjoint to B . Apply Corollary 26.(2) to $B, \gamma_1, \gamma_\mathfrak{c}$ to obtain B_1 . Use Corollary 26.(2) successively to $B_k, \gamma_k, \gamma_\mathfrak{c}$ to obtain a nested sequence of closed disks $\{B_k\}_{k \geq 1}$. Their limit, B' , is cellular and satisfies $B' \cap \tilde{X} = B \cap \tilde{X}$.

Consider, for every k , the subarc $\gamma'_k = \gamma_k|_{[0, t_k]}$ where $t_k = \min\{t : \gamma_k(t) \in B'\}$. Replace $\gamma_\mathfrak{c}$ by a lead line γ that also determines \mathfrak{c} and is composed of two arcs: the first one (possibly empty) is the initial part of some γ'_k , say γ'_{k_0} , while the second subarc is the final part of $\gamma_\mathfrak{c}$ and does not meet any γ'_k for $k \neq k_0$. The index k_0 is easily identified in the prime end closure of \tilde{U} : the lead lines γ_k divide \tilde{U} in infinitely many components and $\gamma_\mathfrak{c}[t, 1]$ is entirely contained in one of them if t is close to 1 because $\mathfrak{c} \neq \mathfrak{c}_k$. Note that the integer k_0 must satisfy $\mathfrak{c}_{k_0+1} \prec \mathfrak{c} \prec \mathfrak{c}_{k_0-1}$.

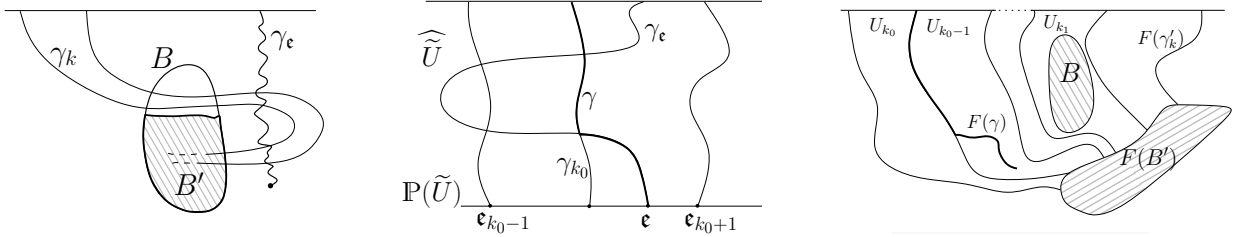


FIGURE 4. Left: construction of B' . Center: Construction of γ seen in the prime end closure of \tilde{U} . Right: Conclusion of the claim in the proof of Proposition 27.

The union of all subarcs $F(\gamma'_k)$ and $F(B')$ separates $\tilde{\mathbb{A}}$ in many connected components. Denote U_k the unique component adherent to $F(\gamma'_k)$ and $F(\gamma'_{k+1})$. Since B is disjoint to $F(B')$ and to all $F(\gamma'_k)$, B meets at most one of these components, say U_{k_1} . $F(\gamma)$ is either contained in $\overline{U_{k_0-1}}$ or in $\overline{U_{k_0}}$. Indeed, $F(\gamma)$ only meets $F(\gamma'_k)$ if $k = k_0$ and is disjoint to $F(B')$.

The only possibility for $F(\gamma)$ to meet B is that $F(\gamma) \subset \overline{U_{k_1}}$ and $B \subset U_{k_1}$, in this case we would have $k_0 \leq k_1 + 1$. If we set $\mathfrak{c} = \mathfrak{c}_{k_1+2}$ in case k_1 is defined (otherwise, any \mathfrak{c} is valid) it follows that $F(\gamma)$ can never meet B . Thus, $\hat{F}(\mathfrak{c}) \in \mathcal{A}^\mathfrak{c}$ and the claim is proven.

Fix $\mathfrak{c} \in \mathcal{P}(\tilde{U})$ so that $\mathcal{A}^\mathfrak{c}$ is positively invariant. Take a lead line $\gamma_\mathfrak{q}$ that determines \mathfrak{q} . By compactness, $\gamma_\mathfrak{q}$ does not meet $T^m(B)$ if m is large enough, for instance $m \geq m^*$. Equivalently, $T^{-m}\mathfrak{q} \in \mathcal{A}^\mathfrak{c}$ for every $m \geq m^*$. Choose an element m_i of the sequence in the statement larger than m^* . Then, $F^{n_i}T^{-m_i}(\mathfrak{q}) \in \mathcal{A}^\mathfrak{c}$ but its principal point $F^{n_i}T^{-m_i}(q) = T^{-m_i}F^{n_i}(q)$ belongs to B , which is absurd. \square

The following lemma eases the conclusion of the subsequent proposition. It is also a sequel of the technical lemma above.

Lemma 28. Let $\mathfrak{e}_-, \mathfrak{e}_+$ be accessible prime ends such that $\mathfrak{e}_- \prec \mathfrak{e}_+$ and their principal points belong to the interior of a closed disk D . Suppose $\mathfrak{e} \in [\mathfrak{e}_-, \mathfrak{e}_+]$ is accessible and $\gamma_{\mathfrak{e}}, \gamma_{\mathfrak{e}_-}, \gamma_{\mathfrak{e}_+}$ are lead lines that determine $\mathfrak{e}, \mathfrak{e}_-, \mathfrak{e}_+$, respectively. If

$$\gamma_{\mathfrak{e}}(1) < \min\{(\gamma_{\mathfrak{e}_-})_1, (\gamma_{\mathfrak{e}_+})_1\} - \text{diam}\{(D)_1\} \text{ or } \gamma_{\mathfrak{e}}(1) > \max\{(\gamma_{\mathfrak{e}_-})_1, (\gamma_{\mathfrak{e}_+})_1\} + \text{diam}\{(D)_1\}$$

then $\gamma_{\mathfrak{e}} \cap D \neq \emptyset$.

Proof. We proceed by contradiction, suppose that $\gamma_{\mathfrak{e}} \cap D = \emptyset$. Assume, without loss of generality, that $\gamma_{\mathfrak{e}_-}$ and $\gamma_{\mathfrak{e}_+}$ are disjoint, intersect $\gamma_{\mathfrak{e}}$ in finitely many points and $\gamma_{\mathfrak{e}_-} \cap D$ and $\gamma_{\mathfrak{e}_+} \cap D$ have finitely many connected components. As a consequence, $\tilde{A} \setminus \tilde{X}$ is divided by $\gamma_{\mathfrak{e}_-} \cup \gamma_{\mathfrak{e}_+}$ into three connected components. Denote V the only component which is relatively compact when seen into the prime end closure, the other two components are adherent to the ends at $-\infty, +\infty$, respectively, of the line of prime ends. Define $t_0 = \max\{t : \gamma_{\mathfrak{e}}(t) \in \gamma_{\mathfrak{e}_-} \cup \gamma_{\mathfrak{e}_+}\}$ or $t_0 = 0$ if the lead lines are disjoint. Clearly, $t_0 < 1$ and, more important, $\gamma_{\mathfrak{e}}(t_0, 1) \subset V$.

Denote $\{\alpha_i^-\}_{i=1}^{n_-}, \{\alpha_i^+\}_{i=1}^{n_+}$ the connected components of $\gamma_{\mathfrak{e}_-} \cap D$ and $\gamma_{\mathfrak{e}_+} \cap D$ other than the ones ending at the landing points and ordered as they are traversed by $\gamma_{\mathfrak{e}_-}, \gamma_{\mathfrak{e}_+}$, respectively. Define k_-, k_+ as the lowest indices for which $\alpha_{k_-}^-, \alpha_{k_+}^+$ satisfy the first alternative in Lemma 25 or set $k_- = n_- + 1, k_+ = n_+ + 1$ if there are none. All the components $\{\alpha_i^-\}_{i=1}^{k_-}, \{\alpha_i^+\}_{i=1}^{k_+}$ bound a negligible piece of D as expressed in the second alternative of Lemma 25. Apply twice Corollary 26.(1) to obtain a closed disk $D' \subset D$ that does not intersect α_i^{\pm} if $i \leq k_{\pm}$ and such that $D' \cap \tilde{X} = D \cap X$.

Set $t_{\pm} = \min\{t : \gamma_{\mathfrak{e}_{\pm}}(t) \in D'\}$ and denote V' the connected component of $V \setminus D'$ adherent to ∂A , and thus to $\gamma_{\mathfrak{e}_-}[0, t_-]$ and to $\gamma_{\mathfrak{e}_+}[0, t_+]$ as well. By the construction of D' , there is a cross-cut c_- of \tilde{U} contained in $\partial D'$ to which $\gamma_{\mathfrak{e}_-}(t_-)$ belongs. Clearly, $\mathfrak{e}_- \in \widehat{V}(c_-)$. The set $\gamma_{\mathfrak{e}_-}[0, t_-] \cup c_-$ divides the prime end closure into three regions: the one enclosed by c_- and the other two W_-, W_+ such that $W_- \cap \mathbb{P}(\tilde{U}) = (-\infty, \mathfrak{e}^*)$ and $W_+ \cap \mathbb{P}(\tilde{U}) = (\mathfrak{e}^{**}, +\infty)$ for $\mathfrak{e}^* \prec \mathfrak{e} \prec \mathfrak{e}^{**}$. Moreover, $W_- \cap V = \emptyset$. Since $\gamma_{\mathfrak{e}}$ does not intersect D , it can not traverse c_- so $\gamma_{\mathfrak{e}} \cap \gamma_{\mathfrak{e}_-} \subset \gamma_{\mathfrak{e}_-}[0, t_-]$. Similarly, we obtain that $\gamma_{\mathfrak{e}} \cap \gamma_{\mathfrak{e}_+} \subset \gamma_{\mathfrak{e}_+}[0, t_+]$. We conclude that $\gamma_{\mathfrak{e}}(t_0, 1) \subset V'$, hence $\gamma_{\mathfrak{e}}(1) \in \overline{V'}$.

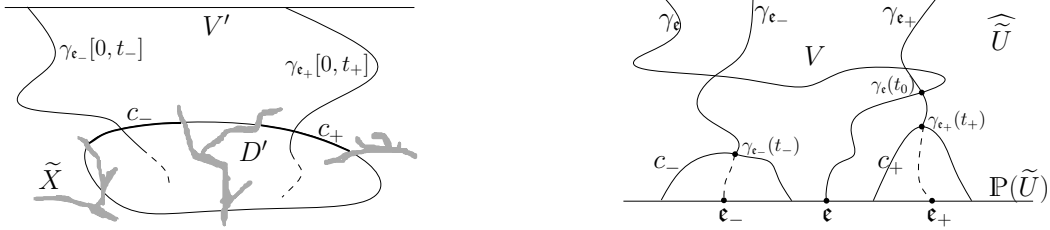


FIGURE 5. Elements of the proof of Lemma 28 in A (left) and in the prime end closure of \tilde{U} (right).

The conclusion now follows from the inclusion $D' \subset D$ and the following observation:

$$(\overline{V'})_1 \subset [\min\{(\gamma_{\mathfrak{e}_-})_1, (\gamma_{\mathfrak{e}_+})_1\} - \text{diam}\{(D')_1\}, \max\{(\gamma_{\mathfrak{e}_-})_1, (\gamma_{\mathfrak{e}_+})_1\} + \text{diam}\{(D')_1\}].$$

□

Proposition 29. Suppose $\rho \leq 0$. Let B be a closed ball satisfying the following properties:

- $B \cap F(B) = \emptyset$.
- There exists a prime end $\mathfrak{c} \in \mathbb{P}(\tilde{U})$ such that if $\mathfrak{e} \prec \mathfrak{c}$, \mathfrak{e} is accessible and its principal point belongs to B then every lead line which determines $\widehat{F}(\mathfrak{e})$ intersects B .
- For every prime end \mathfrak{c}' , there is an accessible prime end $\mathfrak{e}, \mathfrak{e} \prec \mathfrak{c}'$ whose principal point belongs to $\text{int}(B)$.

Then, there is $C > 0$ such that for any accessible point $p \in \tilde{X}$, $|(F^n(p))_1 - (p)_1 - n\rho| < C$ holds for every $n \leq 0$.

Proof. Consider the set \mathcal{A}_B composed of the accessible prime ends $\mathfrak{e} \prec \mathfrak{c}$ for which every lead line that determines \mathfrak{e} meets B . The proof goes as follows: first we prove that \mathcal{A}_B is positively invariant under F and then we find the constant C which makes the inequality hold.

Let us see that $\widehat{F}(\mathcal{A}_B) \subset \mathcal{A}_B$. Assume on the contrary that $\mathfrak{e} \in \mathcal{A}_B, F(\mathfrak{e}) \notin \mathcal{A}_B$ for some prime end \mathfrak{e} . In other words, we can find a lead line γ that determines \mathfrak{e} such that $F(\gamma) \cap B = \emptyset$. By definition of \mathcal{A}_B , γ meets B .

In view of the second hypothesis in the statement, the principal point of \mathfrak{e} does not lie in B . Corollary 26.(3) forces the existence of a cross-cut β of \widetilde{U} contained in ∂B such that $\mathfrak{e} \in \widehat{V}(\beta)$. In fact, \mathfrak{e} belongs to the interior of $\widehat{V}(\beta)$ in the prime end closure.

Denote \mathfrak{e}' the accessible prime end determined by β with $\mathfrak{e}' \prec \mathfrak{e}$. Note that $\mathfrak{e}' \prec \mathfrak{e}$ and its principal point belongs to B . Construct a lead line $\gamma' \subset \gamma \cup \beta$ which determines \mathfrak{e}' . Then, $F(\gamma') \subset F(\gamma) \cup F(\beta) \subset F(\gamma) \cup F(B)$ so $F(\gamma')$ does not meet B even though it determines $\widehat{F}(\mathfrak{e}')$. This contradicts the existence of $\mathfrak{e} \in \mathcal{A}_B \setminus \widehat{F}^{-1}(\mathcal{A}_B)$.

Henceforth, let p be an arbitrary accessible point of \widetilde{X} . Denote \mathfrak{p} a prime end determined by p . Observe first that if m is large enough, say $m \geq m^* > 0$, \mathfrak{p} can be accessed with a lead line that does not meet $T^m B$ or, equivalently, there is a lead line that determines $\widehat{T}^{-m}(\mathfrak{p})$ and does not intersect B , i.e., $\widehat{T}^{-m}(\mathfrak{p}) \notin \mathcal{A}_B$.

The third item in the statement guarantees the existence of two accessible prime ends $\mathfrak{e}_0, \mathfrak{e}_1$ whose principal points lie in B and satisfy $\mathfrak{e}_1 \prec \widehat{T}^{-1}\mathfrak{e}_0 \prec \mathfrak{e}_0 \prec \min\{\mathfrak{e}, \widehat{T}^{-m^*}\mathfrak{p}\}$. For every $n \leq 0$, a translation of $\widehat{F}^n(\mathfrak{p})$, say $\widehat{T}^{-m(n)}\widehat{F}^n(\mathfrak{p})$ that belongs to $[\mathfrak{e}_1, \mathfrak{e}_0]$. Note that for every $n \leq 0$, $m(n) \geq m^*$ because $\widehat{T}^{-m^*}(\mathfrak{p}) \preceq \widehat{T}^{-1}(\mathfrak{p}) \prec \widehat{F}^n(\mathfrak{p})$ since the rotation number is non-positive.

Let $\gamma_{\mathfrak{e}_0}, \gamma_{\mathfrak{e}_1}$ be two lead lines that determine $\mathfrak{e}_0, \mathfrak{e}_1$, respectively. From Lemma 28 we obtain $C' > 0$ so that if $\mathfrak{e} \in [\mathfrak{e}_1, \mathfrak{e}_0]$ is accessible and $(e)_1 \notin [-C', C']$, where e is the principal point of \mathfrak{e} , then any lead line that determines \mathfrak{e} meets B , i.e., $\mathfrak{e} \in \mathcal{A}_B$. This implies, in particular, that $|(T^{-m(0)}(p))_1| = |(p)_1 - m(0)| < C'$ because $\widehat{T}^{-m(0)}(\mathfrak{p}) \notin \mathcal{A}_B$.

For any $n \leq 0$,

$$\begin{aligned} (1) \quad |(F^n(p))_1 - (p)_1 - n\rho| &= |(T^{-m(n)}F^n(p))_1 + m(n) - (p)_1 - n\rho| \leq \\ &\leq |(T^{-m(n)}F^n(p))_1| + |m(n) - (p)_1 - n\rho| \\ &\leq |(T^{-m(n)}F^n(p))_1| + |(p)_1 - m(0)| + |m(n) - m(0) - n\rho| \end{aligned}$$

Note that the rightmost term is bounded by $l + 1$, where l denotes the length of the interval $[\mathfrak{e}_0, \mathfrak{e}_1]$, for all $n \leq 0$. Since $\widehat{T}^{-m(n)}(\mathfrak{p}) \preceq \widehat{T}^{-m^*}(\mathfrak{p})$, we have that $\widehat{T}^{-m(n)}(\mathfrak{p}) \notin \mathcal{A}_B$. From the positive invariance of \mathcal{A}_B we deduce (remember $n \leq 0$) that $\widehat{F}^n\widehat{T}^{-m(n)}(\mathfrak{p}) = \widehat{T}^{-m(n)}\widehat{F}^n(\mathfrak{p}) \notin \mathcal{A}_B$, so $|(T^{-m(n)}F^n(p))_1| \leq C'$. If we set $C = 2C' + l + 1$ then (1) yields $|(F^n(p))_1 - (p)_1 - n\rho| < C$. Since C does not depend on p , the conclusion follows. \square

Proof of Theorem 24 (continuation). The limit point q_0 given by (H2) is not fixed by hypothesis. Thus, as was hinted in advance, we can take B to be a small ball centered at q_0 which is free. We can apply Proposition 27 to obtain \mathfrak{e} and then apply Proposition 29 and conclude the proof. Notice that the third hypothesis in Proposition 29 is a direct consequence of (H2) and our choice of B . \square

Theorem 2.(2b) is then a trivial consequence of Theorem 24. Indeed, if the drifts of a bounded forward semi-orbit and a bounded backward semi-orbit are unbounded at least one hypothesis (H1)–(H2) is not satisfied. However, in the case $\rho \in \mathbb{Q}$ (H1) always holds trivially so (H2) must fail.

As a corollary we deduce a more general variant of Theorem 1. It extends Theorem 2.(2a) to \mathbb{A} .

Theorem 30. *If the sequence $\{(F^n(p))_1 - n\rho\}_{n \leq 0}$ is unbounded for some accessible point $p \in \widetilde{X}$ with bounded backward semi-orbit then $\lim_{n \rightarrow +\infty} (F^n(q))_1/n$ exists and is equal to ρ for every accessible point $q \in \widetilde{X}$ with bounded forward semi-orbit.*

Proof. Let us proceed by contradiction. Assume the result is false for $q \in \widetilde{X}$. Possibly after changing the lift of the angular coordinate by $\tilde{\theta} \mapsto -\tilde{\theta}$ we can assume $\rho < \rho' < \limsup_{n \rightarrow +\infty} (F^n(q))_1/n$, for a fixed $\rho' \in \mathbb{Q}$. Note that this limit always exists and is finite because it is bounded by the maximum of the displacement function on $\mathcal{O}^+(q)$. Lemma 21 guarantees we can apply Theorem 24 and deduce that $\{(F^n(p))_1 - n\rho\}_{n \leq 0}$ is bounded for all accessible points $p \in \widetilde{X}$. \square

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